

# HEDGING OF GAME OPTIONS UNDER MODEL UNCERTAINTY IN DISCRETE TIME

YAN DOLINSKY  
DEPARTMENT OF STATISTICS  
HEBREW UNIVERSITY, JERUSALEM  
ISRAEL

ABSTRACT. We introduce a setup of model uncertainty in discrete time. In this setup we derive dual expressions for the super-replication prices of game options with upper semicontinuous payoffs. We show that the super-replication price is equal to the supremum over a special (non dominated) set of martingale measures, of the corresponding Dynkin games values. This type of results is also new for American options.

## 1. INTRODUCTION

A game contingent claim (GCC) or game option, which was introduced in [10], is defined as a contract between the seller and the buyer of the option such that both have the right to exercise it at any time up to a maturity date (horizon)  $T$ . If the buyer exercises the contract at time  $t$  then he receives the payment  $Y_t$ , but if the seller exercises (cancels) the contract before the buyer then the latter receives  $X_t$ . The difference  $\Delta_t = X_t - Y_t$  is the penalty which the seller pays to the buyer for the contract cancellation. In short, if the seller will exercise at a stopping time  $\sigma \leq T$  and the buyer at a stopping time  $\tau \leq T$  then the former pays to the latter the amount  $H(\sigma, \tau)$  where

$$H(\sigma, \tau) = X_\sigma \mathbb{I}_{\sigma < \tau} + Y_\tau \mathbb{I}_{\tau \leq \sigma}$$

and we set  $\mathbb{I}_Q = 1$  if an event  $Q$  occurs and  $\mathbb{I}_Q = 0$  if not.

A hedge (for the seller) against a GCC is defined as a pair  $(\pi, \sigma)$  that consists of a self financing strategy  $\pi$  and a stopping time  $\sigma$  which is the cancellation time for the seller. A hedge is called perfect if no matter what exercise time the buyer chooses, the seller can cover his liability to the buyer.

Until now there is quite a good understanding of pricing game options in the case where the probabilistic model is given. For details see [11] and the references therein. However, so far super-replication of American options and game options was not studied in the case of volatility uncertainty. In fact, super-replication under volatility uncertainty was studied only for European options (see, [6], [7], [13], [14] and [19]). In the papers (see, [7], [14] and [19]) the authors established a connection between  $G$ -expectation which was introduced by Peng (see [16] and [17]), and super-replication under volatility uncertainty in continuous time models.

---

*Date:* April 15, 2013.

*2010 Mathematics Subject Classification.* Primary: 91G10 Secondary: 60F05, 60G40.

*Key words and phrases.* Dynkin games, game options, super-replication, volatility uncertainty, weak convergence.

In this paper we introduce a discrete setup of volatility uncertainty. We consider a simple model which consists of a savings account and of one risky asset, and we assume that the payoffs are upper semicontinuous. Our main result says that the super-replication price is equal to the supremum over a special (non dominated) set of martingale measures, of the corresponding Dynkin games values. In continuous time models, the problem remains open for American options and game options.

Main results of this paper are formulated in the next section. In Section 3 we prove the main results of the paper for continuous payoffs. This proof is quite elementary and does not use advanced tools. In section 4 we extend the main results for upper semicontinuous payoffs. This extension is technically involved and requires the establishment of some stability results for Dynkin games under weak convergence.

## 2. PRELIMINARIES AND MAIN RESULTS

First we introduce a discrete time version of volatility uncertainty. Let  $N \in \mathbb{N}$ ,  $s > 0$  and  $I = [a, b] \subset \mathbb{R}_+$ . Define the set  $K \subset \mathbb{R}_{++}^{N+1}$  by

$$K = \{(x_0, \dots, x_N) : x_0 = s, \quad |\ln x_{i+1} - \ln x_i| \in I, \quad i < N\}.$$

The financial market consists of a savings account  $B$  and a risky asset  $S$  (stock). The stock price process is  $S_k$ ,  $k = 0, 1, \dots, N$ , where  $N < \infty$  is the maturity date or the total number of allowed trades. By discounting, we normalize  $B \equiv 1$ . We assume that the stock price process satisfies  $(S_0, \dots, S_N) \in K$ . Namely the initial stock price is  $S_0 = s$  and for any  $i < N$  we have  $|\ln S_{i+1} - \ln S_i| \in I$ . This is the only assumption that we make on our financial market and we do not assume any probabilistic structure.

For any  $k = 0, 1, \dots, N$  let  $F_k, G_k : K \rightarrow \mathbb{R}_+$  be upper semicontinuous functions with the following properties, for any  $u, v \in K$ ,  $F_k(u) = F_k(v)$  and  $G_k(u) = G_k(v)$  if  $u_i = v_i$  for all  $i = 0, 1, \dots, k$ . Furthermore, we assume that  $F_k \leq G_k$ .

Consider a game option with the payoff function

$$(2.1) \quad \mathbb{H}(k, l, S) = G_k(S) \mathbb{I}_{k < l} + F_l(S) \mathbb{I}_{l \leq k}, \quad k, l = 0, 1, \dots, N.$$

Observe that  $\mathbb{H}(k, l, S)$  is the reward that the buyer receives given that his exercise time is  $l$  and that the seller cancellation time is  $k$ . Furthermore, the reward  $\mathbb{H}(k, l, S)$  depends only on the stock history up to the moment  $k \wedge l$ .

In our setup a portfolio with initial capital  $x$  is a pair  $\pi = (x, \gamma)$  where  $\gamma : \{0, 1, \dots, N-1\} \times K \rightarrow \mathbb{R}$  is a progressively measurable process, namely for any  $k = 0, 1, \dots, N-1$  and  $u, v \in K$ ,  $\gamma(k, u) = \gamma(k, v)$  if  $u_i = v_i$  for all  $i = 0, 1, \dots, k$ . The portfolio value at time  $k$  is given by

$$(2.2) \quad V_k^\pi(S) = x + \sum_{i=0}^{k-1} \gamma(i, S)(S_{i+1} - S_i), \quad S \in K, \quad k = 0, 1, \dots, N.$$

A stopping time is a measurable function  $\sigma : K \rightarrow \{0, 1, \dots, N\}$  which satisfies the following, for any  $u \in K$  and  $k = 0, 1, \dots, N$  if  $\sigma(u) = k$  then  $\sigma(v) = k$  for any  $v$  with  $v_i = u_i$  for all  $i = 0, 1, \dots, k$ .

A pair  $(\pi, \sigma)$  of a self financing strategy  $\pi$  and a stopping time  $\sigma$  will be called a hedge. A hedge  $(\pi, \sigma)$  will be called perfect if

$$(2.3) \quad V_{\sigma(S) \wedge l}^\pi(S) \geq \mathbb{H}(\sigma(S), l, S), \quad \forall S \in K, \quad l = 0, 1, \dots, N.$$

The super-replication price is given by

$$(2.4) \quad \mathbb{V} = \inf \{V_0^\pi \mid \text{there exists a stopping time } \sigma \text{ such that } (\pi, \sigma) \text{ is a perfect hedge}\}.$$

Observe that we do not have any underlying probability measure, and we require to construct a super-hedge for any possible values of the stock prices. Similar setup (but not the same) was studied in [6] for European options.

We make some preparations before we formulate the main result of the paper. Let  $Z = (Z_0, \dots, Z_N)$  be the canonical process on the Euclidean space  $\mathbb{R}^{N+1}$ . Namely for any  $z = (z_0, \dots, z_N) \in \mathbb{R}^{N+1}$  and  $k \leq N$  we have  $Z_k(z) = z_k$ . A probability measure  $\mathbb{P}$  supported on  $K$  is called a martingale law if for any  $k < N$

$$(2.5) \quad \mathbb{E}_{\mathbb{P}}(Z_N | Z_0, \dots, Z_k) = Z_k \quad \mathbb{P} \text{ a.s.}$$

where  $\mathbb{E}_{\mathbb{P}}$  denotes the expectation with respect to  $\mathbb{P}$ . Denote by  $\mathcal{M}$  the set of all martingale laws. Clearly,  $\mathcal{M} \neq \emptyset$ . For instance the probability measure  $\mathbb{P}_b$  which is given by

$$\mathbb{P}_b(Z_0 = s) = 1 \quad \text{and}$$

$$\mathbb{P}_b(\ln Z_{i+1} - \ln Z_i = b) = 1 - \mathbb{P}_b(\ln Z_{i+1} - \ln Z_i = -b) = \frac{1-e^{-b}}{e^b - e^{-b}}, \quad i < N,$$

is an element in  $\mathcal{M}$ .

Let  $\mathcal{F}_k = \sigma(Z_0, \dots, Z_k)$ ,  $k \leq N$  be the canonical filtration, and let  $\mathcal{T}$  be the set of all stopping times (with respect to the above filtration) with values in the set  $\{0, 1, \dots, N\}$ .

The following theorem is the main result of the paper.

**Theorem 2.1.** *The super-replication price is given by*

$$\begin{aligned} \mathbb{V} &= \inf_{\sigma \in \mathcal{T}} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) = \\ &= \sup_{\mathbb{P} \in \mathcal{M}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) = \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z). \end{aligned}$$

It is well known that  $\inf \sup \geq \sup \inf$ , thus in order to prove Theorem 2.1 it is sufficient to prove the following relations

$$(2.6) \quad \mathbb{V} \leq \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z)$$

and

$$(2.7) \quad \mathbb{V} \geq \inf_{\sigma \in \mathcal{T}} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z).$$

The first inequality is the difficult one and it will be proved in Sections 3–4. The second inequality is simpler and we show it by the following argument.

From (2.4) it follows that for any  $\epsilon > 0$  there exists a perfect hedge  $(\tilde{\pi}, \tilde{\sigma})$  with an initial capital  $V_0^{\tilde{\pi}} = \mathbb{V} + \epsilon$ . From (2.2) we get that for any  $\mathbb{P} \in \mathcal{M}$  the stochastic process  $\{V_k^{\tilde{\pi}}(Z)\}_{k=0}^N$  is a martingale with respect to  $\mathbb{P}$ . Observe that  $\tilde{\sigma}(Z) \in \mathcal{T}$ , and so from (2.3) we obtain that for any  $\tau \in \mathcal{T}$

$$\mathbb{V} + \epsilon = V_0^{\tilde{\pi}} = \mathbb{E}_{\mathbb{P}} V_{\tilde{\sigma}(Z) \wedge \tau}^{\tilde{\pi}} \geq \mathbb{E}_{\mathbb{P}} \mathbb{H}(\tilde{\sigma}(Z), \tau, Z).$$

The terms  $\mathbb{P} \in \mathcal{M}$  and  $\tau \in \mathcal{T}$  are arbitrary, thus we conclude that

$$\mathbb{V} + \epsilon \geq \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\tilde{\sigma}(Z), \tau, Z) \geq \inf_{\sigma \in \mathcal{T}} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z).$$

By letting  $\epsilon \downarrow 0$  we derive (2.7). □

**Remark 2.2.** From Theorem 2.1 we obtain the following probabilistic corollary.

$$\inf_{\sigma \in \mathcal{T}} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) = \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z).$$

This corollary is not obvious since the set  $\mathcal{M}$  is a set of non dominated probability measures, and so it does not follow from the results in [12].

### 3. PROOF OF THE MAIN RESULT

This section is devoted to the proof of (2.6), for the case where the functions  $F_k, G_k : K \rightarrow \mathbb{R}_+, k \leq N$  are continuous.

**3.1. Discretization of the space.** Let  $n \in \mathbb{N}$ . Introduce the set

$$K_n := \{(x_0, \dots, x_N) : x_0 = s \text{ and } |\ln x_{i+1} - \ln x_i| \in \{a, a + (b-a)/n, a + 2(b-a)/n, \dots, b\}\}.$$

Consider a multinomial model for which the stock price  $S = (S_0, \dots, S_N)$  lies in the set  $K_n$ . As before the savings account is given by  $B \equiv 1$ . In this model a portfolio with an initial capital  $x$  is a pair  $\pi = (x, \gamma)$  where  $\gamma : \{0, 1, \dots, N-1\} \times K_n \rightarrow \mathbb{R}$  is a progressively measurable process. A hedge is a pair  $(\pi, \sigma)$  which consists of a portfolio strategy  $\pi$  and a stopping time  $\sigma$ . A stopping time is a map  $\sigma : K_n \rightarrow \{0, 1, \dots, N\}$  which satisfies that if  $\sigma(u) = k$  then  $\sigma(v) = k$  for any  $v$  with  $v_i = u_i$  for all  $i = 0, 1, \dots, k$ . A hedge  $(\pi, \sigma)$  will be called perfect if

$$(3.1) \quad V_{\sigma(S) \wedge l}^{\pi}(S) \geq \mathbb{H}(\sigma(S), l, S), \quad \forall S \in K_n, \quad l = 0, 1, \dots, N$$

where the portfolio value is given by the same formula as (2.2).

Let

$$(3.2) \quad \mathbb{V}_n = \inf \{V_0^{\pi} \mid \text{there exists a stopping time } \sigma \text{ such that } (\pi, \sigma) \text{ is a perfect hedge}\}$$

be the super-replication price in the multinomial model. Next, we introduce a modified super-replication price. Let  $M > 0$  and let  $\Gamma_M$  be the set of all portfolio strategies  $\pi = (x, \gamma)$  where  $\gamma : \{0, 1, \dots, N-1\} \times K_n \rightarrow [-M, M]$ . Namely, we consider portfolios for which the absolute value of the number of stocks is not exceeding  $M$ . Consider the super-replication price

$$\mathbb{V}_n^M = \inf_{\pi \in \Gamma_M} \{V_0^{\pi} \mid \text{there exists a stopping time } \sigma \text{ such that } (\pi, \sigma) \text{ is a perfect hedge}\}.$$

We will need the following technical lemma.

**Lemma 3.1.** *There exists a constant  $M > 0$  (which is independent of  $n$ ) such that*

$$\mathbb{V}_n^M = \mathbb{V}_n.$$

*Proof.* Clearly,  $\mathbb{V}_n^M \geq \mathbb{V}_n$ . Thus it is sufficient to show that  $\mathbb{V}_n^M \leq \mathbb{V}_n$ . Set

$$A = \max_{0 \leq k \leq N} \sup_{x \in K} F_k(x).$$

Clearly there exists a perfect hedge with an initial capital  $A$  (in this case the investor does not trade and stop only at the maturity). Let  $(\pi, \sigma)$  be a perfect hedge in the sense of (3.1). We will assume (without loss of generality) that the initial capital  $V_0^{\pi}$  is no bigger than  $A > 0$ . Furthermore, since the option is exercised no later than in the moment  $\sigma(S)$ , we can assume (without loss of generality) that  $\gamma(k, S) \equiv 0$  for  $k \geq \sigma(S)$ .

First let us prove by induction that for any  $S \in K_n$  and  $k = 0, 1, \dots, N$ ,

$$(3.3) \quad V_{k \wedge \sigma(S)}^\pi(S) \leq A \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^k \quad \text{and} \quad |\gamma(k, S)| \leq \frac{A \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^k}{(1 - e^{-b})S_k}.$$

If  $\sigma \equiv 0$  then the statement is clear. Thus we assume that  $\sigma(S) > 0$  for any ( $\sigma$  is a stopping time)  $S \in K_n$ . Choose  $S \in K_n$ . Clearly, the portfolio value at time 1 should be non negative, for any possible growth rate of the stock. In particular we have,

$$V_0^\pi(S) + \gamma(0, S)s(e^b - 1) \geq 0 \quad \text{and} \quad V_0^\pi(S) + \gamma(0, S)s(e^{-b} - 1) \geq 0$$

and we conclude that  $|\gamma(0, S)| \leq \frac{A}{s(1 - e^{-b})}$ . Thus (3.1) holds for  $k = 0$ . Next, assume that (3.3) holds for  $k$ , and we prove it for  $k + 1$ . From the induction assumption we get

$$\begin{aligned} V_{(k+1) \wedge \sigma(S)}^\pi(S) &= V_{k \wedge \sigma(S)}^\pi(S) + \gamma(k \wedge \sigma(S), S)(S_{(k+1) \wedge \sigma(S)} - S_{k \wedge \sigma(S)}) \leq \\ &A \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^k + \frac{A \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^k}{(1 - e^{-b})S_k} S_k(e^b - 1) \leq A \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^{k+1}, \end{aligned}$$

as required. Next, if  $\sigma(S) \leq k + 1$  then  $\gamma(k + 1, S) = 0$ . If  $\sigma(S) > k + 1$ , then the portfolio value at time  $k + 2$  should be non negative, for any possible growth rate of the stock. Thus,

$$V_{k+1}^\pi(S) + \gamma(k + 1, S)S_{k+1}(e^b - 1) \geq 0 \quad \text{and} \quad V_{k+1}^\pi(S) + \gamma(k + 1, S)S_{k+1}(e^{-b} - 1) \geq 0$$

and so,

$$|\gamma(k + 1, S)| \leq \frac{V_{k+1}^\pi(S)}{(1 - e^{-b})S_{k+1}} \leq \frac{A \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^{k+1}}{(1 - e^{-b})S_{k+1}}.$$

This completes the proof of (3.3). Finally, observe that  $S_k \geq se^{-bk}$  and so, we conclude that for  $M := As \frac{e^{bN}}{1 - e^{-b}} \left( 1 + \frac{e^b - 1}{1 - e^{-b}} \right)^N$ , we have  $|\gamma(k, S)| \leq M$  for all  $k, S$ .  $\square$

Now, we can easily prove the following lemma.

**Lemma 3.2.**

$$\mathbb{V} \leq \liminf_{n \rightarrow \infty} \mathbb{V}_n.$$

*Proof.* Fix  $\epsilon > 0$ . Let  $n \in \mathbb{N}$ . Consider the multinomial model for which the stock price process  $S = (S_0, \dots, S_N)$  lies in the set  $K_n$ . Let  $(\pi, \sigma)$  be a perfect hedge for this multinomial model such that  $\pi = (\mathbb{V}_n + \epsilon, \gamma)$ . From lemma 3.1 it follows that we can assume that  $|\gamma(k, S)| \leq M$  for any  $k, S$ . Consider the map  $\psi_n : K \rightarrow K_n$  which is given by  $\psi_n(y_0, \dots, y_N) = (x_0, \dots, x_N)$  where

$$\begin{aligned} x_0 &= y_0 \quad \text{and for } k > 0 \quad \ln y_{i+1} = \ln y_i \\ &+ \text{sgn}(\ln x_{i+1} - \ln x_i)(a + (b - a)[n(|\ln x_{i+1} - \ln x_i| - a)/(b - a)]/n) \end{aligned}$$

where  $[v]$  is the integer part of  $v$  and  $\text{sgn}(v) = 1$  for  $v > 0$  and  $= -1$  otherwise. For the original financial market define a hedge  $(\tilde{\pi}, \tilde{\sigma})$  by the following relations,  $\tilde{\pi} = (\mathbb{V}_n + 2\epsilon, \tilde{\gamma})$  where

$$(3.4) \quad \tilde{\gamma}(k, S) = \gamma(k, \psi_n(S)) \quad \text{and} \quad \tilde{\sigma}(S) = \sigma(\psi_n(S)), \quad k < N, \quad S \in K.$$

Observe that  $\tilde{\gamma}$  is a progressively measurable map and  $\tilde{\sigma}$  is a stopping time. Thus  $(\tilde{\pi}, \tilde{\sigma})$  is indeed a hedge for the original financial market. From the continuity of the functions  $F_k, G_k, k = 0, 1, \dots, N$  it follows that for sufficiently large  $n$

$$(3.5) \quad \|S - \psi_n(S)\| + |F_k(S) - F_k(\psi_n(S))| + |G_k(S) - G_k(\psi_n(S))| < \frac{\epsilon}{2MN}, \quad S \in K, k \leq N,$$

where we denote  $\|(z_0, \dots, z_N)\| = \max_{0 \leq i \leq N} |z_i|$ . Let  $S \in K$ . Set  $Y^{(n)} = \psi_n(S)$ . From (3.5) and the fact that  $\gamma \in [-M, M]$  it follows that (for sufficiently large  $n$ ) for any  $l \leq N$  we get

$$\begin{aligned} V_{l \wedge \tilde{\sigma}(S)}^{\tilde{\pi}}(S) &= \epsilon + V_{l \wedge \sigma(Y^{(n)})}^{\pi}(Y^{(n)}) + \\ &\sum_{k=0}^{l \wedge \tilde{\sigma}(S)-1} \gamma(k, Y^{(n)})((S_{k+1} - S_k) - (Y_{k+1}^{(n)} - Y_k^{(n)})) \geq \\ &\epsilon + \mathbb{H}(\sigma(Y^{(n)}), l, Y^{(n)}) - 2NM\|S - Y^{(n)}\| \geq \mathbb{H}(\tilde{\sigma}(S), l, S). \end{aligned}$$

Thus for sufficiently large  $n$ ,  $\mathbb{V} \leq 2\epsilon + \mathbb{V}_n$ . Since  $\epsilon > 0$  was arbitrary this concludes the proof.  $\square$

**3.2. Analysis of the multinomial models.** Fix  $n \in \mathbb{N}$ . Let  $\Omega = \mathbb{R}^{N+1}$ . Define the piecewise constant stochastic processes

$$\begin{aligned} S_t^{(n)}(z_0, \dots, z_N) &:= z_{[nt]}, \quad Y_t^{(n)}(z_0, \dots, z_N) = F_{[nt]}(z_0, \dots, z_N) \\ \text{and } X_t^{(n)} &= G_{[nt]}(z_0, \dots, z_N), \quad z \in \Omega, t \in [0, 1]. \end{aligned}$$

Let  $\{\mathcal{F}_t^{(n)}\}_{t=0}^1$  be the filtration which is generated by the process  $S^{(n)}$ . The set  $K_n \subset \Omega$  is finite, and so, there exists a probability measure  $\mathbb{P}_n$  on  $\Omega$  which is supported on  $K_n$  and gives to any element in  $K_n$  a positive probability. Thus we can apply Theorem 2.2 in [12] for a market with one risky asset  $S^{(n)}$  which lives on the probability space  $(\Omega, \mathcal{F}_1^{(n)}, \mathbb{P}_n)$ , and a game option with the payoffs  $Y^{(n)} \leq X^{(n)}$ . In this case the super-replication price coincides with  $\mathbb{V}_n$  which is given by (3.2). Thus let  $\mathcal{M}_n \subset \mathcal{M}$  be the set of all martingale laws which are supported on the set  $K_n$  and  $\mathbb{T}$  be the set of all stopping times (with respect to the filtration  $\{\mathcal{F}_t^{(n)}\}_{t=0}^1$ ) with values in the set  $[0, 1]$ . From Theorem 2.2 in [12] and the fact that the processes  $Y^{(n)}, X^{(n)}$  are piecewise constant we obtain

$$\mathbb{V}_n = \sup_{\mathbb{P} \in \mathcal{M}_n} \sup_{\tau \in \mathbb{T}} \inf_{\sigma \in \mathbb{T}} \mathbb{E}_{\mathbb{P}}(X_{\sigma}^{(n)} \mathbb{I}_{\sigma < \tau} + Y_{\tau}^{(n)} \mathbb{I}_{\sigma \geq \tau}) = \sup_{\mathbb{P} \in \mathcal{M}_n} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z).$$

Since  $\mathcal{M}_n \subset \mathcal{M}$ , we conclude that for any  $n \in \mathbb{N}$ ,

$$\mathbb{V}_n \leq \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z).$$

This together with Lemma 3.2 completes the proof of (2.6).  $\square$

**Remark 3.3.** *An interesting question which remains open is the limit behavior where the maturity date  $N$  goes to infinity. Namely, for a given  $N \in \mathbb{N}$  consider the interval  $I := I(N) = \left[\frac{a}{\sqrt{N}}, \frac{b}{\sqrt{N}}\right]$ . Our conjecture is that for regular enough payoffs the limit behavior of the super-replication prices  $\mathbb{V} := \mathbb{V}(N)$  as  $N \rightarrow \infty$  is equal to a stochastic game version of  $G$ -expectation, defined on the canonical space  $\mathcal{C}[0, T]$ . For European options the limit is the standard  $G$ -expectation, this follows from [5] and [9]. It seems that the tool which was employed in [5] can work for the American options case. In this case the limit of the super-replication prices is*

equal to an optimal stopping version of  $G$ -expectation. However for game options the problem is more complicated.

#### 4. EXTENSION FOR UPPER SEMICONTINUOUS PAYOFFS

In this section we prove (2.6) for the case where the functions  $F_k, G_k : K \rightarrow \mathbb{R}_+$ ,  $k \leq N$  are upper semicontinuous (and not necessarily continuous).

Let  $\mathcal{A} = \max_{0 \leq k \leq N} \sup_{x \in K} G_k(x) < \infty$ . By using similar arguments as in Lemma 5.3 in [8] it follows that for any  $k = 0, 1, \dots, N$  there are two sequences of continuous functions  $\{F_k^{(n)}\}_{n=1}^\infty$  and  $\{G_k^{(n)}\}_{n=1}^\infty$  which satisfy the following:

- (i).  $\mathcal{A} \geq G_k^{(n)} \geq G_k$ ,  $\mathcal{A} \geq F_k^{(n)} \geq F_k$  and  $G_k^{(n)} \geq F_k^{(n)}$ , for all  $n$ .
- (ii).

$$(4.1) \quad \limsup_{n \rightarrow \infty} G_k^{(n)}(x_n) \leq G^{(n)}(x) \quad \text{and} \quad \limsup_{n \rightarrow \infty} F_k^{(n)}(x_n) \leq F(x)$$

for every  $x \in K$  and every sequence  $\{x_n\}_{n=1}^\infty \subset K$  with  $\lim_{n \rightarrow \infty} x_n = x$ .

- (iii). Furthermore, for any  $n \in \mathbb{N}$  and  $u, v \in K$ ,  $F_k^{(n)}(u) = F_k^{(n)}(v)$  and  $G_k^{(n)}(u) = G_k^{(n)}(v)$  if  $u_i = v_i$  for all  $i = 0, 1, \dots, k$ .

Let  $\mathbb{V}$  be the super-replication price which corresponds to the payoff functions  $F, G$ , and for any  $n \in \mathbb{N}$  let  $\mathbb{V}_n$  be the super-replication price which corresponds to the payoff functions  $F^{(n)}, G^{(n)}$ .

From (i), it follows that for any  $n \in \mathbb{N}$ ,  $\mathbb{V} \leq \mathbb{V}_n$ . Thus from Theorem 2.1 (for continuous payoffs) it follows that

$$\mathbb{V} \leq \liminf_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}^{(n)}(\sigma, \tau, Z)$$

where

$$\mathbb{H}^{(n)}(k, l, S) = G_k^{(n)}(S) \mathbb{I}_{k < l} + F_l^{(n)}(S) \mathbb{I}_{l \leq k}, \quad k, l = 0, 1, \dots, N, \quad S \in K.$$

We conclude that in order to establish (2.6) we need to prove the following lemma.

**Lemma 4.1.**

$$\sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) = \liminf_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}^{(n)}(\sigma, \tau, Z).$$

*Proof.* From (i),

$$\sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) \leq \liminf_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}^{(n)}(\sigma, \tau, Z).$$

Thus we will prove that (infact this is the inequality that we need)

$$(4.2) \quad \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) \geq \liminf_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}^{(n)}(\sigma, \tau, Z).$$

For any  $n \in \mathbb{N}$ , let  $\mathbb{P}_n \in \mathcal{M}$  and  $\rho_n \in \mathcal{T}$  be such that

$$(4.3) \quad \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}^{(n)}(\sigma, \tau, Z) < \frac{1}{n} + \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}_n} \mathbb{H}^{(n)}(\sigma, \rho_n, Z).$$

Consider the set  $\Pi$  of all probability measures on  $K$ , induced with the topology of weak convergence. Observe that  $\Pi$  is a compact set (this follows from Prohorov's theorem, see [2] Section 1 for details). From the existence of the regular distribution function (for details see [18] page 227) we obtain that there exist measurable functions  $h_k^{(n)} : \mathbb{R}^{k+1} \rightarrow \Pi$ ,  $k < N$ , such that for any Borel set  $A \subset K$  and  $n \in \mathbb{N}$

$$\mathbb{P}_n((Z_0, \dots, Z_N) \in A | Z_0, \dots, Z_k) = h_k^{(n)}(Z_0, \dots, Z_k)(A), \quad \mathbb{P}_n \text{ a.s.}$$

For any  $n \in \mathbb{N}$  consider the distribution of (under the measure  $\mathbb{P}_n$ )

$$(\rho_n, Z_0, \dots, Z_N, h_0^{(n)}(Z_0), \dots, h_{N-1}^{(n)}(Z_0, \dots, Z_{N-1}))$$

on the space  $[0, N] \times K \times \Pi^N$  with the product topology.

Since the space  $[0, N] \times K \times \Pi^N$  is compact then by Prohorov's theorem there is a subsequence which for simplicity we still denote by

$$(\rho_n, Z_0, \dots, Z_N, h_0^{(n)}(Z_0), \dots, h_{N-1}^{(n)}(Z_0, \dots, Z_{N-1})), \quad n \in \mathbb{N}$$

which converges weakly. Thus from the Skorohod representation theorem (see [3]) we obtain that we can redefine the sequence

$$(\rho_n, Z_0, \dots, Z_N, h_0^{(n)}(Z_0), \dots, h_{N-1}^{(n)}(Z_0, \dots, Z_{N-1})), \quad n \in \mathbb{N}$$

on a probability space  $(\Omega, \mathcal{F}, P)$  such that we have  $P$  a.s convergence

$$(\rho_n, Z_0^{(n)}, \dots, Z_N^{(n)}, h_0^{(n)}(Z_0^{(n)}), \dots, h_{N-1}^{(n)}(Z_0^{(n)}, \dots, Z_{N-1}^{(n)})) \rightarrow (\rho, U_0, \dots, U_N, W_1, \dots, W_N).$$

Redefining means that for any  $n \in \mathbb{N}$  the distribution of

$$(\rho_n, Z_0, \dots, Z_N, h_0^{(n)}(Z_0), \dots, h_{N-1}^{(n)}(Z_0, \dots, Z_{N-1}))$$

under  $\mathbb{P}_n$  is equal to the distribution of

$$(\rho_n, Z_0^{(n)}, \dots, Z_N^{(n)}, h_0^{(n)}(Z_0^{(n)}), \dots, h_{N-1}^{(n)}(Z_0^{(n)}, \dots, Z_{N-1}^{(n)}))$$

under  $P$ . Let  $\mathcal{G}_k = \sigma\{U_0, \dots, U_k\}$ ,  $k \leq N$  be the filtration which is generated by  $U_0, \dots, U_N$ . Denote by  $\mathcal{T}_U$  the set of all stopping times (with respect to this filtration) with values in the set  $\{0, 1, \dots, N\}$ . Next we show the following three properties:

- (I). The distribution of  $(U_0, \dots, U_N)$  (on the space  $K$ ) is an element in  $\mathcal{M}$ .
- (II). For any  $k$ , the conditional distribution of  $(U_0, \dots, U_N)$  given  $U_0, \dots, U_k$  equals to  $W_k$ .
- (III). For any  $k$ , the event  $\{\tau = k\}$  and  $\mathcal{G}_N$  are independent given  $\mathcal{G}_k$ .

Denote by  $E$  the expectation with respect to  $P$ . From Lebesgue's dominated convergence theorem it follows that for any  $k \leq N$  and continuous bounded functions  $f : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ ,  $g : K \rightarrow \mathbb{R}$  we have

$$\begin{aligned} (4.5) \quad & E((U_N - U_k)f(U_0, \dots, U_k)) = \\ & \lim_{n \rightarrow \infty} E((Z_N^{(n)} - Z_k^{(n)})f(Z_0^{(n)}, \dots, Z_k^{(n)})) \\ & = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_n}((Z_N - Z_k)f(Z_0, \dots, Z_k)) = 0, \end{aligned}$$

where the last equality follows the fact that  $\mathbb{P}_n \in \mathcal{M}$  is a martingale distribution. From the definition of the topology on  $\Pi$ , we also have

$$\begin{aligned} (4.6) \quad & E(f(U_0, \dots, U_k)g(U_0, \dots, U_N)) = \\ & \lim_{n \rightarrow \infty} E(f(Z_0^{(n)}, \dots, Z_k^{(n)})g(Z_0^{(n)}, \dots, Z_N^{(n)})) \\ & = \lim_{n \rightarrow \infty} E(f(Z_0^{(n)}, \dots, Z_k^{(n)}) \int g(y)h_k^{(n)}(Z_0^{(n)}, \dots, Z_k^{(n)})(dy)) \\ & = E(f(U_0, \dots, U_k) \int g(y)W_k(dy)). \end{aligned}$$

By applying standard density arguments we obtain that (4.5) implies (I) and (4.6) implies (II). Next, fix  $k$ . From (II) and the fact that  $\rho_n$  is a stopping time we obtain

that

$$\begin{aligned}
E(\mathbb{I}_{\rho=k} E(g(U_0, \dots, U_N) | \mathcal{G}_k)) &= E(\mathbb{I}_{\rho=k} \int g(y) W_k(dy)) = \\
\lim_{n \rightarrow \infty} E(\mathbb{I}_{\rho_n=k} \int g(y) h_k^{(n)}(Z_0^{(n)}, \dots, Z_k^{(n)})(dy)) &= \\
\lim_{n \rightarrow \infty} E(\mathbb{I}_{\rho_n=k} E(g(Z_0^{(n)}, \dots, Z_N^{(n)}) | Z_0^{(n)}, \dots, Z_k^{(n)})) &= \\
\lim_{n \rightarrow \infty} E(g(Z_0^{(n)}, \dots, Z_N^{(n)}) \mathbb{I}_{\rho_n=k}) &= E(g(U_0, \dots, U_N) \mathbb{I}_{\rho=k}),
\end{aligned}$$

and again, from standard density arguments we conclude that

$$E(\mathbb{I}_{\rho=k} | \mathcal{G}_k) = E(\mathbb{I}_{\rho=k} | \mathcal{G}_N)$$

and (III) follows. Property (III) is important because it implies the following. For any stochastic process  $(L_0, \dots, L_N)$  which is adapted to the filtration  $\mathcal{G}_k$ ,  $k \leq N$ , we have

$$(4.7) \quad EL_\rho \leq \sup_{\tau \in \mathcal{T}_U} EL_\tau.$$

The proof of this implication can be done in the same way as in Lemma 3.3 in [4], and so we omit it.

Now we arrive to the final step of the proof. Choose  $0 < \epsilon < 1$ . Let  $\tilde{\sigma} \in \mathcal{T}_U$  be such that

$$(4.8) \quad \inf_{\sigma \in \mathcal{T}_U} E\mathbb{H}(\sigma, \rho, U) > E\mathbb{H}(\tilde{\sigma}, \rho, U) - \epsilon,$$

where  $U = (U_0, \dots, U_N)$ . For any  $k$  there exists a continuous function  $f_k : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  such that  $P(\mathbb{I}_{\tilde{\sigma}=k} \neq f_k(U_0, \dots, U_k)) < \frac{\epsilon}{2^{k+1}}$ . For any  $n \in \mathbb{N}$  define  $\tilde{\sigma}_n = N \wedge \min\{k | f_k(Z_0^{(n)}, \dots, Z_k^{(n)}) > \frac{1}{2}\}$ . Clearly  $\tilde{\sigma}_n$  is a stopping time with respect to the filtration generated by  $Z_0^{(n)}, \dots, Z_N^{(n)}$ . Let  $C$  be the following set

$$C = \{\omega \in \Omega | \exists m := m(\omega) \text{ such that } \forall n > m \tilde{\sigma}_n(\omega) = \tilde{\sigma}(\omega)\}.$$

From (4.4) and the fact that  $f_k$ ,  $k \leq N$  are continuous functions, it follows that

$$(4.9) \quad P(C) \geq 1 - \sum_{i=0}^N \frac{\epsilon}{2^{i+1}} \geq 1 - \epsilon.$$

Observe that (4.4) also implies that a.s.  $\rho_n(\omega) = \rho(\omega)$  for sufficiently large  $n$  (which depends on  $\omega$ ). Thus from property (ii) we get

$$\mathbb{H}(\tilde{\sigma}, \rho, U) \mathbb{I}_C \geq \limsup_{n \rightarrow \infty} \mathbb{H}^{(n)}(\tilde{\sigma}_n, \rho_n, Z^{(n)}) \mathbb{I}_C$$

where  $Z^{(n)} = (Z_0^{(n)}, \dots, Z_N^{(n)})$ . Since  $\mathbb{H}$  and  $\mathbb{H}^{(n)}$  are uniformly bounded by  $\mathcal{A}$  then from Fatou's lemma we derive

$$(4.10) \quad E\mathbb{H}(\tilde{\sigma}, \rho, U) \mathbb{I}_C \geq \limsup_{n \rightarrow \infty} E\mathbb{H}^{(n)}(\tilde{\sigma}_n, \rho_n, Z^{(n)}) \mathbb{I}_C.$$

Finally, let  $\mathcal{Q}$  be the distribution of  $(U_0, \dots, U_N)$ . From (I) it follows that  $\mathcal{Q} \in \mathcal{M}$  is a martingale distribution. It is well known that for Dynkin games the inf and the sup can be exchanged (for details see [15]). Thus from (4.3)–(4.4), (4.7) for

$L_k = H(\sigma, k, U)$  and (4.8)–(4.10) we get

$$\begin{aligned}
& \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}(\sigma, \tau, Z) \geq \sup_{\tau \in \mathcal{T}_U} \inf_{\sigma \in \mathcal{T}_U} E\mathbb{H}(\sigma, \tau, U) \\
& = \inf_{\sigma \in \mathcal{T}_U} \sup_{\tau \in \mathcal{T}_U} E\mathbb{H}(\sigma, \tau, U) \geq \inf_{\sigma \in \mathcal{T}_U} E\mathbb{H}(\sigma, \rho, U) \\
& \geq E\mathbb{H}(\tilde{\sigma}, \rho, U) \mathbb{I}_C - \epsilon \geq \limsup_{n \rightarrow \infty} E\mathbb{H}^{(n)}(\tilde{\sigma}_n, \rho_n, Z^{(n)}) \mathbb{I}_C - \epsilon \\
& \geq \limsup_{n \rightarrow \infty} E\mathbb{H}^{(n)}(\tilde{\sigma}_n, \rho_n, Z^{(n)}) - \mathcal{A}\epsilon - \epsilon \\
& \geq \limsup_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \mathbb{H}^{(n)}(\sigma, \tau, Z) - \epsilon(\mathcal{A} + 1),
\end{aligned}$$

and since  $\epsilon$  was arbitrary we obtain (4.2) as required. The reason that we have  $\limsup$  in the above equation and not  $\liminf$  as in (4.2), is because we passed to a subsequence, but left the same notations.  $\square$

**Remark 4.2.** *Let us notice that in order to obtain Lemma 4.1 we used a stronger form of the standard weak convergence. Namely we also required a convergence of the conditional distributions. This is the discrete analog of the extended weak convergence which introduced by Aldous in [1] for continuous time processes. In general, the standard weak convergence is not sufficient for the convergence of the corresponding optimal stopping and Dynkin games values.*

## REFERENCES

- [1] D.Aldous, *Weak convergence of stochastic processes for processes viewed in the strasbourg manner*, Unpublished Manuscript, Statis. Laboratory Univ. Cambridge, (1981).
- [2] P.Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.
- [3] R.M.Dudley, Distances of Probability Measures and Random Variables, *Ann. Math. Statist.* **39**, 1563–1572, (1968).
- [4] Y.Dolinsky, *Applications of Weak Convergence for Hedging of Game Options*, *Ann.Appl. Probab.* **20**, 1891–1906, (2010).
- [5] Y.Dolinsky, *Numerical Schemes for G-Expectations*, *Electronic Journal of Probability*, **17**, (2012).
- [6] S. Deparis and C. Martini, *Superhedging Strategies and Balayage in Discrete Time*, *Proceedings of the 4th Ascona Conference on Stochastic Analysis*, (2004).
- [7] L. Denis and C. Martini, *A Theoretical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty*, *Ann. Appl. Probab.* **16**, 827–852, (2006).
- [8] Y. Dolinsky and H.M. Soner, *Robust Hedging with the Proportional Transaction Costs*, submitted.
- [9] Y. Dolinsky, M. Nutz and H.M. Soner, *Weak Approximations of G-Expectations*, *Stochastic Processes and their Applications*, **2**, 664–675, (2012).
- [10] Yu. Kifer, *Game options*, *Finance and Stoch.* **4**, 443–463, (2000).
- [11] Yu. Kifer, *Dynkin games and Israeli options*, *ISRN Probability and Statistics*, to appear.
- [12] J. Kallsen and C. Kuhn, *Convertible bonds: Financial derivatives of game type*, *Exotic Option Pricing and Advanced Levy Models*, pages 277–291. Wiley, New York, 2005.
- [13] M.Nutz, *Superreplication under Model Uncertainty in Discrete Time*, submitted.
- [14] M. Nutz, H.M. Soner, *Superhedging and Dynamic Risk Measures under Volatility Uncertainty*, *SIAM Journal on Control and Optimization*, **50**, 2065–2089, (2012).
- [15] Y.Ohtsubo, *Optimal stopping in sequential games with or without a constraint of always terminating*, *Math. Oper. Res.* **11** (1986), 591–607.
- [16] S.Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Itô type*, *Stochastic Analysis and Applications*, volume 2 of *Abel Symp.*, (2007), 541–567, Springer Berlin.
- [17] S.Peng, *Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation.*, *Stochastic. Processes. Appl.*, **12** (2008), 2223–2253.
- [18] A.N.Shiryaev, *Probability*, Springer-Verlag, New York, (1984).
- [19] H.M.Soner, N.Touzi, and J.Zhang, *Martingale representation theorem for the G-expectation*, *Stochastic. Processes. Appl.*, **2** (2011), 265–287.

DEPARTMENT OF STATISTICS, HEBREW UNIVERSITY, MOUNT SCOPUS, JERUSALEM 91905, E.MAIL:  
YAN.DOLINSKY@MAIL.HUJI.AC.IL